Continuity of a percolation function on the hierarchical group

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Abstract

We consider a long-range percolation in the hierarchical group Ω_N of order N where probability of connection between two nodes separated by distance k is of the form $\min\{\alpha\beta^{-k}, 1\}$, $\alpha \geq 0$ and $\beta > 0$. The percolation function $\theta(\alpha, \beta)$ is defined as the probability of having a infinite component contain the origin $\mathbf{0} \in \Omega_N$. We show that $\theta(\alpha, \beta)$ is continuous with respect to both α and β.

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1 Introduction

In 1957, Broadbent and Hammersley first systematically studied the percolation theory in the Euclidean lattice $\mathbb{Z}^d$. The infinity of the space of vertices and its geometry are principal features of this model; see e.g. [10] for back ground. Some questions of percolation in other non-Euclidean infinite systems is formulated in [4]. The study of long-range percolation on $\mathbb{Z}^d$ traces back to [14] and leads to a range of interesting results in probability theory and statistical physics [1, 5, 6, 7, 16, 17, 19]. On the other hand, hierarchical structures have been used in applications in the physics, genetics and social sciences thanks to the multi-scale organization of many natural objects [3, 12].

Recently, long-range percolation is investigated on the hierarchical group Ω_N of order N (to be defined below), where classical methods for the usual lattice break down. The

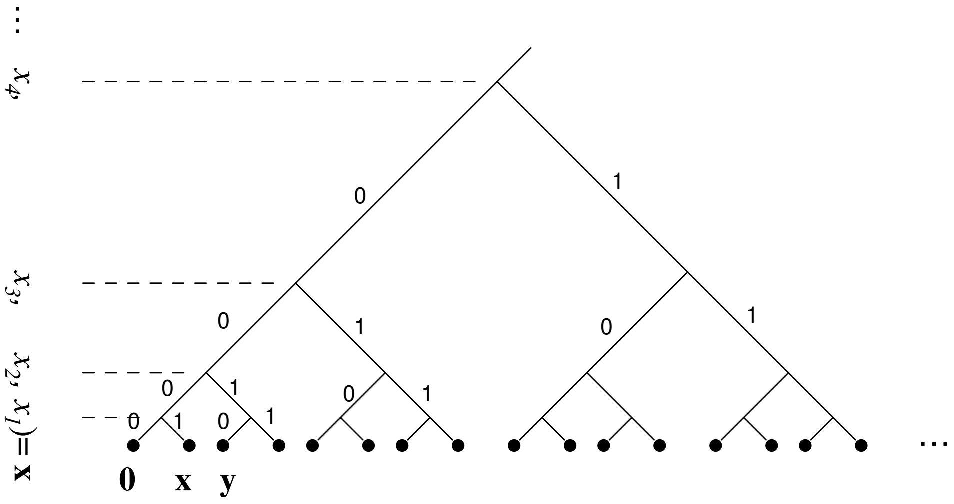

Figure 1: An illustration of hierarchical group Ω_2 of order 2. The distances between three vertices $\mathbf{0} = (0, 0, 0, \cdots)$, $\mathbf{x} = (1, 0, 0, \cdots)$ and $\mathbf{y} = (0, 1, 0, \cdots)$ are $d(\mathbf{0}, \mathbf{x}) = 1$ and $d(\mathbf{0}, \mathbf{y}) = d(\mathbf{x}, \mathbf{y}) = 2$.

asymptotic long-range percolation on Ω_N is addressed in [9] for $N \to \infty$. The work [8, 11, 15] analyze the phase transition of long-range percolation on Ω_N for finite N using different connection probabilities and methodologies. The contact process on Ω_N for fixed N has been investigated in [2]. In this paper, we consider another important question: the continuity of percolation functions of the percolation on Ω_N for fixed N.

For an integer $N \geq 2$, we define the set

$$\Omega_N := \left\{ \mathbf{x} = (x_1, x_2, \cdots) : x_i \in \{0, 1, \cdots, N-1\}, i = 1, 2, \cdots, \sum_{i=1}^{\infty} x_i < \infty \right\}, \tag{1}$$

and define a metric d on it:

$$d(\mathbf{x}, \mathbf{y}) = \begin{cases} 0, & \mathbf{x} = \mathbf{y}, \\ \max\{i : x_i \neq y_i\}, & \mathbf{x} \neq \mathbf{y}. \end{cases} \tag{2}$$

The pair (Ω_N, d) is referred to as the hierarchical group (or hierarchical lattice) of order N, which may be thought of as the set of leaves at the bottom of an infinite regular tree without a root, where the distance between two vertices is the number of levels (generations) from the bottom to their most recent common ancestor. Figure 1 shows the group Ω_2 along with its metric generating tree.

Such a distance d satisfies the strong triangle inequality

$$d(\mathbf{x}, \mathbf{y}) \leq \max\{d(\mathbf{x}, \mathbf{z}), d(\mathbf{z}, \mathbf{y})\}, \tag{3}$$

for any triple $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \Omega_N$. Hence, (Ω_N, d) is an ultrametric (or non-Archimedean) space [13]. From its ultrametricity, it is clear that for every $\mathbf{x} \in \Omega_N$ there are $(N-1)N^{k-1}$ vertices at distance k from it.

Now consider a long-range percolation on Ω_N. For each $k \geq 1$, the probability of connection between $\mathbf{x}$ and $\mathbf{y}$ such that $d(\mathbf{x}, \mathbf{y}) = k$ is given by

$$p_k = \min\left\{\frac{\alpha}{\beta^k}, 1\right\}, \tag{4}$$

where $0 \leq \alpha < \infty$ and $0 < \beta < \infty$, all connections being independent. Two vertices $\mathbf{x}, \mathbf{y} \in \Omega_N$ are in the same component (or cluster) if there exists a finite sequence $\mathbf{x} = \mathbf{x}_0, \mathbf{x}_1, \cdots, \mathbf{x}_n = \mathbf{y}$ of vertices such that each pair $(\mathbf{x}_{i-1}, \mathbf{x}_i)$, $i = 1, \cdots, n$, of vertices presents an edge.

The rest of the paper is organized as follows. Section 2 contains the main results and Section 3 is devoted to the proofs.

2 Main results

Let $\mathbb{N}$ be the non-negative integers including 0, and denote by $\ell := \min\{k \in \mathbb{N} : \alpha \leq \beta^{k+1}\}$. Let $|S|$ be the size of a set S. The connected component containing the node $\mathbf{x} \in \Omega_N$ is denoted by $C(\mathbf{x})$. Since, for every node $\mathbf{x}$, $|C(\mathbf{x})|$ has the same distribution, it suffices to consider only $|C(\mathbf{0})|$. The percolation probability is defined as

$$\theta(\alpha, \beta) := P_{\alpha,\beta}(|C(\mathbf{0})| = \infty), \tag{5}$$

where $P_{\alpha,\beta}$ is the probability measure governing this percolation process. We may omit the subscripts α and β whenever possible. The critical percolation value is defined as

$$\alpha_c(\beta) := \inf\{\alpha \geq 0 : \theta(\alpha, \beta) > 0\}. \tag{6}$$

The following theorem characterizes the phase transition for this model.

Theorem 1. ([15])

(i) If $\beta \leq N$, then $\theta(\alpha, \beta) \equiv 1$, and hence $\alpha_c(\beta) = 0$;

(ii) If $N < \beta < N^2$, then $0 < \alpha_c(\beta) < \infty$;

(iii) If $\beta \geq N^2$, then $\theta(\alpha, \beta) \equiv 0$, and hence $\alpha_c(\beta) = \infty$.

The uniqueness of infinite component and large component size are established in the following results.

Theorem 2. ([16]) *For $0 \leq \alpha < \infty$ and $0 < \beta < \infty$, there is at most one infinite component almost surely.*

Theorem 3. ([18]) *Suppose that α and β are such that $\theta := \theta(\alpha, \beta) > 0$, i.e., $0 < \beta < N^2$. Therefore, for every $\varepsilon > 0$,*

$$\lim_{k\to\infty} P(|C_k^m(\mathbf{0})| > (\theta - \varepsilon)N^k) = 1. \tag{7}$$

Before presenting our main result, we give some notations. For any vertex $\mathbf{x} \in \Omega_N$, define $B_r(\mathbf{x})$ the ball of radius r around $\mathbf{x}$, that is, $B_r(\mathbf{x}) = \{\mathbf{y} : d(\mathbf{x}, \mathbf{y}) \leq r\}$. From this definition we make the following observations. Firstly, for any $\mathbf{x} \in \Omega_N$, $B_r(\mathbf{x})$ contains N^r vertices. Secondly, $B_r(\mathbf{x}) = B_r(\mathbf{y})$ if $d(\mathbf{x}, \mathbf{y}) \leq r$. Finally, for any $\mathbf{x}$, $\mathbf{y}$ and r, we either have $B_r(\mathbf{x}) = B_r(\mathbf{y})$ or $B_r(\mathbf{x}) \cap B_r(\mathbf{y}) = \emptyset$.

For a set S of vertices, denote by $\bar{S} = \Omega_N \backslash S$ its complement. Let $C_n(\mathbf{x})$ be the cluster of vertices that are connected to $\mathbf{x}$ by a path using only vertices within $B_n(\mathbf{x})$. For disjoint sets $S_1, S_2 \subseteq \Omega_N$, we denote by $S_1 \leftrightarrow S_2$ the event that at least one edge joins a vertex in S_1 to a vertex in S_2. $S_1 \not\leftrightarrow S_2$ means the event that such an edge does not exist. Let $C_n^m(\mathbf{x})$ be the largest clusters in $B_n(\mathbf{x})$. If there are more than one such clusters, just take any one of them as $C_n^m(\mathbf{x})$. It is clear that $|C_n^m(\mathbf{x})| = \max_{\mathbf{y}\in B_n(\mathbf{x})} |C_n(\mathbf{y})|$. Our main result is the following.

Theorem 4. *The percolation function $\theta(\alpha, \beta)$ is continuous with respect to both $\alpha \in (0, \infty)$ and $\beta \in (0, \infty)$.*

3 Proof of Theorem 4

In this section, we will prove Theorem 4 through a series of lemmas.

First note that by virtue of Theorem 1, the percolation function $\theta(\alpha, \beta)$ is trivially continuous with respect to $\alpha \in (0, \infty)$ and $\beta \in (0, N] \cup [N^2, \infty)$.

Lemma 1. *$\theta(\alpha, \beta)$ is continuous from the right with respect to $\alpha \in (0, \infty)$ and from left with respect to $\beta \in (N, N^2)$.*

Proof. From the hierarchial structure of Ω_N, we have

$$\{C_i(\mathbf{0}) \leftrightarrow \overline{B_i(\mathbf{0})}\} \subseteq \{C_{i-1}(\mathbf{0}) \leftrightarrow \overline{B_{i-1}(\mathbf{0})}\} \tag{8}$$

and

$$\{|C(\mathbf{0})| = \infty\} = \cap_{i=0}^{\infty}\{C_i(\mathbf{0}) \leftrightarrow \overline{B_i(\mathbf{0})}\}. \tag{9}$$

On the other hand, we obtain

$$\begin{aligned} P(C_i(\mathbf{0}) \leftrightarrow \overline{B_i(\mathbf{0})}) &= E\left(E\left(1_{\{C_i(\mathbf{0})\leftrightarrow\overline{B_i(\mathbf{0})}\}}\,\middle|\, C_i(\mathbf{0})\right)\right) \\ &= E\left(1-\prod_{k=1}^{\infty}(1-p_{i+k})^{|C_i(\mathbf{0})|(N-1)N^{i+k-1}}\right) \\ &= E\left(1-\prod_{k=1}^{\infty}\left(1-\min\left\{\frac{\alpha}{\beta^{i+k}},1\right\}\right)^{|C_i(\mathbf{0})|(N-1)N^{i+k-1}}\right). \end{aligned} \tag{10}$$

If $\ell \geq i$, we have $P(C_i(\mathbf{0}) \leftrightarrow \overline{B_i(\mathbf{0})}) = 1$ from (10). If $\ell < i$, we have

$$P(C_i(\mathbf{0}) \leftrightarrow \overline{B_i(\mathbf{0})}) = E\left(1-\prod_{k=1}^{\infty}\left(1-\frac{\alpha}{\beta^{i+k}}\right)^{|C_i(\mathbf{0})|(N-1)N^{i+k-1}}\right), \tag{11}$$

where the infinite multiplication is convergent when $\beta > N$; c.f. [15].

Since $|C_i(0)|$ depends only on the states of finitely many edges, the expectation in (11) is continuous with respect to α and β. In particular, $P(C_i(\mathbf{0}) \leftrightarrow \overline{B_i(\mathbf{0})})$ is continuous from the left in β for $\beta > N$. Note that $P(C_i(\mathbf{0}) \leftrightarrow \overline{B_i(\mathbf{0})})$ is increasing with respect to α, decreasing with respect to β and decreasing with respect to i. The statement follows from the fact that a decreasing limit of increasing (resp. decreasing) functions, which are continuous from the right (resp. left) is still continuous from the right (resp. left). □

It remains to prove that $\theta(\alpha, \beta)$ is continuous from the left with respect to $\alpha > 0$ and continuous from the right with respect to $\beta \in (N, N^2)$.

Consider a site-bond percolation (i.e., mixed percolation). A measure $P_{\alpha,\beta,\gamma}$ is constructed as follows. A vertex in Ω_N is open with probability $1 - \gamma$, independently of the states of the other vertices in Ω_N. If $\mathbf{x}$ and $\mathbf{y}$ are both open vertices then they are connected with probability $\min\{\alpha\beta^{-d(\mathbf{x},\mathbf{y})}, 1\}$, independently of whether other edges are present or not. By a coupling argument, the following lemma holds.

Lemma 2. *Assume that $\beta > N$. For every $\varepsilon > 0$, there exists $\gamma > 0$ such that*

$$P_{\alpha,\beta}(|C(\mathbf{0})| = \infty) \leq P_{\alpha(1+\varepsilon),\beta,\gamma}(|C(\mathbf{0})| = \infty). \tag{12}$$

Proof. We first construct a directed mixed percolation model on Ω_N; c.f. [11]. Vertices in Ω_N are open with probability $1 - \gamma$. If vertex $\mathbf{x}$ is open, then a directed edge from $\mathbf{x}$ to $\mathbf{y}$ is present with probability $\min\{\alpha\beta^{-d(\mathbf{x},\mathbf{y})}, 1\}$. Conditioned on the states of the vertices the presence or absence of an edge is independent of the presence or absence of other edges. We denote this measure by $\hat{P}_{\alpha,\beta,\gamma}$. The set of vertices which can be reached by a path from vertex $\mathbf{x}$ is denoted by $\hat{C}(\mathbf{x})$. Similarly, we can define the directed version of the original measure as $\hat{P}_{\alpha,\beta}$. We have $\hat{P}_{\alpha,\beta} = \hat{P}_{\alpha,\beta,0}$. It is easy to see that [10]

$$P_{\alpha,\beta,\gamma}(|C(\mathbf{0})| = \infty) = \hat{P}_{\alpha,\beta,\gamma}(|\hat{C}(\mathbf{0})| = \infty). \tag{13}$$

The directed mixed percolation graph with parameters α, β and γ can be obtained as follows. We assign i.i.d. random variables $\{X_{\mathbf{x}}\}$ to the vertices $\{\mathbf{x} \in \Omega_N\}$, satisfying that the probability of not having an edge from $\mathbf{x}$ to $\mathbf{y}$ is equal to $\max\{\alpha\beta^{-r}, 1\} - \alpha\beta^{-r}$ if $d(\mathbf{x}, \mathbf{y}) = r$. We construct a directed multi-graph as follows. Vertices are open with probability $1 - \gamma$, independently of each other. If $\mathbf{x}$ is open, then $X_{\mathbf{x}}$ directed edges emit from $\mathbf{x}$. The endpoints of these edges are independently chosen from $\Omega_N \backslash \mathbf{x}$. If $\mathbf{x}$ is closed, then no edges emit from $\mathbf{x}$. Since

$$P(\mathbf{x} \leftrightarrow \mathbf{y}) = 1 - \left(\max\left\{\frac{\alpha}{\beta^r}, 1\right\} - \frac{\alpha}{\beta^r}\right) = \min\left\{\frac{\alpha}{\beta^r}, 1\right\}, \tag{14}$$

we obtain the original directed graph by replacing the collection of multiple edges from $\mathbf{x}$ to $\mathbf{y}$ by a single edge from $\mathbf{x}$ to $\mathbf{y}$ for all $\mathbf{x}, \mathbf{y} \in \Omega_N$.

Let Z_1 be a random variable with the same distribution as $X_{\mathbf{x}}$. Define a random variable $X_{\mathbf{x}}(\varepsilon)$ which satisfies the fact that the probability of not having an edge from $\mathbf{x}$ to $\mathbf{y}$ is equal to $\max\{\alpha(1+\varepsilon)\beta^{-r}, 1\} - \alpha(1+\varepsilon)\beta^{-r}$ if $d(\mathbf{x}, \mathbf{y}) = r$. Clearly, $X_{\mathbf{x}} = X_{\mathbf{x}}(0)$. Let $Z_2 = Y_1 Y_2$, where Y_1 is equal to 1 with probability $1 - \gamma$ and equal to 0 with probability γ, Y_2 is independent of Y_1, and Y_2 has the same distribution as $X_{\mathbf{x}}(\varepsilon)$. In the statement of Lemma 2, for the ordinary percolation model the number of edges starting at $\mathbf{x}$ in the multi-graph is distributed as Z_1, while for the mixed percolation model, the number of edges starting at $\mathbf{x}$ is distributed as Z_2. For each $\varepsilon > 0$, there is a $\gamma > 0$ such that $P(Z_1 = 0) = P(Z_2 = 0)$. By suitably defining the distributions of $X_{\mathbf{x}}$ and $X(\varepsilon)$, we have

$$\begin{aligned} P(Z_2 > k | Z_2 > 0) &= P(Y_2 > k | Y_1 = 1, Y_2 > 0) \\ &= P(Y_2 > k | Y_2 > 0) \\ &> P(Z_1 > k | Z_1 > 0) \end{aligned} \tag{15}$$

for any $k > 0$. A standard coupling argument then yields (12). □

Employing Lemma 2, we can prove the following result, which together with Lemma 1 implies Theorem 4 immediately.

Lemma 3. *Suppose that $\alpha \in (0,\infty)$ and $\beta \in (N, N^2)$, that is, $\theta(\alpha,\beta) > 0$. For every $0 < \varepsilon < \theta(\alpha,\beta)$, there exists $\delta > 0$ such that $\theta(\alpha,\beta) > \theta(\alpha-\delta,\beta+\delta) > \theta(\alpha,\beta)-\varepsilon$.*

Proof. Fix α,β and $\varepsilon > 0$. The first inequality $\theta(\alpha,\beta) > \theta(\alpha-\delta,\beta+\delta)$ is clear. From Theorem1, Lemma 2 and the fact that for every $\beta \in (N,N^2)$ $\theta(\alpha,\beta) > 1-\varepsilon$ for α large enough [15], we may choose α' such that

$$\theta(\alpha', (N^2+\beta)/2) > 1-\varepsilon/3, \tag{16}$$

and $\gamma \in (0, \varepsilon/3)$ such that

$$\theta(\alpha', (N^2+\beta)/2) \le P_{2\alpha',(N^2+\beta)/2,\gamma}(|C(\mathbf{0})| = \infty). \tag{17}$$

From Theorem 3 and $\beta \in (N,N^2)$, we may choose K large enough such that

$$\left(1-\min\left\{\frac{\alpha-\delta}{\beta^{K+l}},1\right\}\right)^{(\theta(\alpha,\beta)-\varepsilon/3)^2N^{2K}} < 1-\min\left\{\frac{2\alpha'}{\beta^l},1\right\}. \tag{18}$$

In addition, we choose $\delta > 0$ such that $\delta < \min\{\alpha/3, (N^2-\beta)/2\}$ and

$$P_{\alpha-\delta,\beta+\delta}(|C_K^m(\mathbf{0})| > (\theta(\alpha,\beta)-\varepsilon/3)N^K) > 1-\gamma, \tag{19}$$

which is possible in view of the continuity of the probability in α and β for finite K. Note that $(N^2+\beta)/2 > \delta+\beta$.

The ball $B_K(\mathbf{x})$ is said to be good if $C_K^m(\mathbf{x})$ has size more than $(\theta(\alpha,\beta)-\varepsilon/3)N^K$. As in [11], we delete all vertices that are in a ball of radius K which is not good and also all vertices that are not in the largest cluster of good balls. Vertices in the new model are connected if and only if the largest clusters in the original K-balls are connected by an edge. We interpret the remaining components as the vertices of the hierarchical group of order N in which vertices are independently deleted with probability less than γ by (19). Remaining components in the original graph, of which the vertices are at distance $K+l$, are connected by at least one edge with probability more than

$$1-\left(1-\min\left\{\frac{\alpha-\delta}{\beta^{K+l}},1\right\}\right)^{(\theta(\alpha,\beta)-\varepsilon/3)^2N^{2K}} > 1-\left(1-\min\left\{\frac{2\alpha'}{\beta^l},1\right\}\right) \tag{20}$$

using (18) and $\alpha-\delta > 2\alpha/3$, regardless of the existence or absence of other connections. Therefore, the rescaled percolation process stochastically dominates a mixed percolation process with parameters $2\alpha', \beta$ and γ.

By exchangeability and (19), we have

$$
\begin{aligned}
& P_{\alpha-\delta,\beta+\delta}(|C_K(\mathbf{0})| \geq (\theta(\alpha,\beta)-\varepsilon/3)N^K) \\
\geq\ & (\theta(\alpha,\beta)-\varepsilon/3)P_{\alpha-\delta,\beta+\delta}(|C_K^m(\mathbf{0})| > (\theta(\alpha,\beta)-\varepsilon/3)N^K) \\
\geq\ & (\theta(\alpha,\beta)-\varepsilon/3)(1-\gamma). \qquad (21)
\end{aligned}
$$

Conditioned on the origin $\mathbf{0}$ being in the largest component of a good ball, the probability that $\mathbf{0}$ is in an infinite component if the parameters are $\alpha-\delta$ and $\beta+\delta$ is larger than $1-\varepsilon/3$ by using (16) and (17). Combining the above comments and $\gamma < \varepsilon/3$, we obtain

$$
\begin{aligned}
\theta(\alpha-\delta,\beta+\delta) &> (\theta(\alpha,\beta)-\varepsilon/3)(1-\gamma)(1-\varepsilon/3) \\
&> (\theta(\alpha,\beta)-\varepsilon/3)(1-\varepsilon/3)^2 \\
&> \theta(\alpha,\beta)-\varepsilon. \qquad (22)
\end{aligned}
$$

The proof is thus complete. □

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